PTEP
Prog. Theor. Exp. Phys. **2015**, 11B101 (20 pages)
DOI: 10.1093/ptep/ptv033

Structure of Supersymmetric Gauge Theories

The localization principle in SUSY gauge theories

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Received February 10, 2015; Accepted February 13, 2015; Published September 9, 2015

The localization principle is a powerful analytic tool in supersymmetric gauge theories which enables one to perform supersymmetric path integrals explicitly. Many important formulae have been obtained, and they led to a major breakthrough in the understanding of gauge theories at strong coupling as well as the dynamics of branes in M-theory. Some of those results are reviewed, focusing especially on Pestun's solution to four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories on S^4 and the subsequent developments on three- or four-dimensional gauge theories on spheres.

Subject Index B16

1. Introduction

Localization is a powerful mathematical principle that sometimes allows us to reduce the difficulty of integrals over complicated spaces. If a continuous symmetry acts on the space, one can express certain integrals over that space as sums of contributions from fixed points, that is, the points which are invariant under the symmetry. It has been applied to the problems in supersymmetric gauge theories in different ways, and led to a number of useful formulae that can probe the strong coupling dynamics of gauge theories.

1.1. *What is the localization principle?*

Let us explain the basic idea of the localization principle, quoting an illustrative example of the volume of a sphere from [1]. We use the standard polar coordinates θ, ϕ on the sphere S^2, in terms of which the simplectic volume form is given by $\omega = \sin\theta d\theta d\phi$. Using the rotational symmetry generated by the vector field $v = \partial_\phi$, one can think of a deformation of the ordinary derivative d into an equivariant derivative $\mathbf{Q} \equiv d - \epsilon i_v$, and accordingly deform the ordinary closed forms into the differential forms annihilated by $\mathbf{Q}$, called equivariantly closed forms. The volume form ω is then modified into $\omega + \epsilon H$, where $H = \cos\theta$ is the Hamiltonian function for the isometry v. The symplectic volume of S^2 then receives the following modification,

$$4\pi = \int \omega = \int e^{\omega} \quad \Longrightarrow \quad \int e^{\omega+\epsilon H} = \frac{2\pi}{\epsilon}\left(e^{\epsilon} - e^{-\epsilon}\right). \tag{1}$$

Interestingly, in the rightmost expression the two terms can be interpreted as contributions from two fixed points, namely the north and the south poles. Indeed, one can "approximate" the contribution

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from the north pole $\theta = 0$ by a Gaussian integral over local Cartesian coordinates x, y,

$$\int dx\, dy e^{\epsilon\left\{1-\frac{1}{2}\left(x^2+y^2\right)\right\}} = \frac{2\pi e^{\epsilon}}{\epsilon}, \tag{2}$$

where e^{ϵ} is the classical value of $e^{\epsilon H}$ at the north pole and $2\pi/\epsilon$ is the result of Gaussian integration over x, y. The same approximation at the south pole and suitable Wick rotation of the integration contour can explain the other term. This example is the simplest application of the localization principle or the Duistermaat–Heckman formula in mathematics.

In string theory or quantum field theories, complicated spaces often arise as moduli space of solutions to some field equations such as the Bogomol'nyi–Prasad–Sommerfield (BPS) conditions in supersymmetric models. The integrals over such moduli spaces often provide a useful low-energy approximation to the original path integral. Actually, in some supersymmetric theories one can deform the theory in a suitable manner so that the moduli space approximation becomes exact. An example of such deformations is the topological twist of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories, which was invented for studying the cohomology of instanton moduli spaces within the framework of quantum field theory [2]. Another example is the topological A-twist of 2D supersymmetric sigma models and the corresponding topological string, which involve integrals over moduli space of holomorphic maps from Riemann surfaces to Calabi–Yau manifolds [3].

In the above examples, the reduction from infinite-dimensional path integrals to finite-dimensional integrals makes use of the idea of localization based on a fermionic symmetry (supersymmetry) $\mathbf{Q}$. The supersymmetry means the action functional S is invariant under $\mathbf{Q}$, and also the path integral measure is such that the expectation values of $\mathbf{Q}$-exact observables all vanish:

$$\langle \mathbf{Q}(\cdots)\rangle = \int e^{-S}\mathbf{Q}(\cdots) = \int \mathbf{Q}\left(e^{-S}\cdots\right) = 0. \tag{3}$$

These imply that the values of supersymmetric observables do not change under deformations of the theory of the form $S \to S + t\mathbf{Q}\mathcal{V}$ for arbitrary parameter t and fermionic functional $\mathcal{V}$ such that $\mathbf{Q}^2\mathcal{V} = 0$. The supersymmetric path integrals thus localize to *saddle points* characterized by the BPS-like condition

$$\mathbf{Q}\Psi = 0 \quad \text{for all the fermions } \Psi. \tag{4}$$

To evaluate the contribution from each saddle point, one only needs to path integrate over fluctuations with Gaussian approximation, keeping only terms in $\mathbf{Q}\mathcal{V}$ up to the second order in the fluctuations as was done in (2). This gives an exact answer because the supersymmetric observables are t-independent.

Thus, in supersymmetric theories, localization is applied for two different purposes. One is the reduction from an infinite-dimensional path integral to a finite-dimensional integral over moduli spaces (called *SUSY localization* in this article), and the other is the simplification of integrals over complicated moduli spaces using symmetry (*equivariant localization*). The underlying principle is the same: in particular they are both characterized by a fermionic operator $\mathbf{Q}$ which squares to a bosonic symmetry of the system.

1.2. Localization in SUSY gauge theories

The two kinds of localization both played important role in [4] where Nekrasov proposed the topologically twisted gauge theory on an Omega background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$. In this theory, the supercharge $\mathbf{Q}$

squares into a spacetime rotation plus a constant gauge rotation,

$$\mathbf{Q}^2 = \epsilon_1 J_{12} + \epsilon_2 J_{34} + \text{gauge}(a), \tag{5}$$

where a is the expectation value of the scalar field in a vector multiplet which parametrizes the Coulomb branch moduli space. The path integral of this theory defines the so-called Nekrasov instanton partition function, which is the generating function for equivariant integrals over instanton moduli spaces. The parameters ϵ_1, ϵ_2, a play the same role as that of ϵ in (1), and simplify the integrals over instanton moduli spaces to combinatoric sums. Nekrasov's partition function is known to contain the information on the low-energy effective prepotential; in fact there is an extensive study showing it encodes even richer information on the mathematical structure underlying 4D $\mathcal{N} = 2$ supersymmetric gauge theories. See [5] for a review on this field.

In 2007, the idea of SUSY localization was first applied to gauge theories which are not topological field theories. In [6] Pestun used the localization principle to obtain an exact formula for supersymmetric observables in 4D $\mathcal{N} = 2$ SUSY gauge theories on the sphere S^4. He showed that the infinite-dimensional path integral can be reduced to a finite-dimensional integral over Lie algebra, and using the result he gave an analytic proof of the long-standing conjecture about the Wilson loops in $\mathcal{N} = 4$ super Yang–Mills theories [7,8]. In 2009, another exact formulae was found for 3D superconformal Chern–Simons matter theories by Kapustin, Willett, and Yaakov [9]. Together with the application of localization to the 3D superconformal index by Seok Kim [10], these works brought the power of localization to the attention of many physicists.

This article is a brief review of the pioneering work in [6] and [9] and the subsequent developments in supersymmetric gauge theories based on the localization principle. In the first part we will focus mostly on theories in and three dimensions and the developments around exact partition functions on the sphere. In the latter part we will discuss interesting developments regarding supersymmetric deformation of the round sphere called squashings.

1.3. Remark

SUSY localization reduces the path integral to an integral over the space of saddle points, and allows us to treat the fluctuations around saddle points by Gaussian approximation. The Gaussian integral in field theory gives rise to determinants of Laplace or Dirac operators, which are usually defined as infinite products over eigenvalues. In the following we will see many formulae for the determinants. On the face of it those infinite product formulae do not make sense or are simply diverging, but they do make sense by a suitable regularization. Let us not worry too much about the regularization issue; instead, recall that the same kind of infinite product arises even for the path-integral evaluation of the partition function for a single harmonic oscillator:

$$\int \mathcal{D}q(t) \exp\left[-\frac{1}{\hbar}\int_0^\beta dt \left(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2\right)\right] = (\text{const}) \cdot \prod_{n\in\mathbb{Z}} \frac{1}{\beta\omega + 2\pi i n}. \tag{6}$$

The infinite product is understood as the result of path integration over Fourier modes of the periodic variable $q(t) \sim q(t+\beta)$. It needs an appropriate regularization so as to reproduce the desired result, $1/2\sinh(\beta\omega/2)$.

2. 4D $\mathcal{N} = 2$ gauge theories

For 4D $\mathcal{N} = 2$ supersymmetric gauge theories, the exact partition function was obtained for topologically twisted theories on an Omega background in [4]. Based on this result, Pestun [6] obtained the closed formula for supersymmetric observables on S^4. A little later there was a development in the construction and classification of superconformal theories based on the picture of wrapped M5-branes, which led to a new understanding of the relation between $\mathcal{N}=2$ gauge theories and the geometry of Riemann surface [11]. These developments also led to the discovery of a surprising relation between observables in 4D gauge theories and 2D conformal field theories [12,13].

2.1. *Exact solution on S^4*

Let us begin by reviewing the exact results for the theories on S^4. In [6], the theories were constructed by using the conformal map from flat $\mathbb{R}^4$. The supersymmetry is characterized by conformal Killing spinors $\xi_{\alpha A}$, $\bar{\xi}^{\dot{\alpha}}_A$ satisfying

$$\begin{aligned} D_m\xi_A &\equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\sigma_{ab}\right)\xi_A = -i\sigma_m\bar{\xi}'_A, \quad D_m\bar{\xi}'_A = -\frac{i}{4\ell^2}\bar{\sigma}_m\xi_A, \\ D_m\bar{\xi}_A &\equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\bar{\sigma}_{ab}\right)\bar{\xi}_A = -i\bar{\sigma}_m\xi'_A, \quad D_m\xi'_A = -\frac{i}{4\ell^2}\sigma_m\bar{\xi}_A. \end{aligned} \tag{7}$$

The indices α, $\dot{\alpha}$ (usually suppressed) represent spinors under a four-dimensional rotation group, whereas the index A is for doublets under $SU(2)$ R-symmetry. See [14] for the convention of spinor calculus used here. In theories with rigid supersymmetry on curved spaces such as spheres, these Killing spinors appear in SUSY transformation rules in place of constant spinor parameters. Unlike the supersymmetry parameters for theories on flat $\mathbb{R}^4$ they are in general not constant. But they take a fixed form once the diffeomorphism and other local gauge invariance are fixed.

$\mathcal{N}=2$ theory has two supermultiplets. The vector multiplet consists of a vector A_m, a complex scalar ϕ, gauginos $\lambda_{\alpha A}$, $\bar{\lambda}_{\dot{\alpha}A}$, and auxiliary scalar fields. The hypermultiplet consists of an $SU(2)_{\mathrm{R}}$ doublet scalar q_A, fermions ψ_α, $\bar{\psi}_{\dot{\alpha}}$, and auxiliary fields. Once the gauge group G and representation R for the hypermultiplet are specified, one can construct a supersymmetric Laglangian for the vector multiplet,

$$S_{\mathrm{YM}} = \int d^4x\sqrt{g}\,\mathrm{Tr}\left(\frac{1}{2g^2}F_{mn}F^{mn} + \frac{i\theta}{32\pi^2}\varepsilon^{klmn}F_{kl}F_{mn} + \cdots\right), \tag{8}$$

and the kinetic Lagrangian for hypermultiplets coupled to vector multiplets. One can also include the hypermultiplet mass term (or other SUSY invariant called FI term which we will not discuss here) in the action. The partition function will then be a function of the gauge coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, the matter mass m, and the radius ℓ of the sphere.

The existence of SUSY theories on spheres was known and even used in the study of superconformal indices or construction of superstring worldsheet theories. But the notion of conformal Killing spinors and the fully explicit construction of supersymmetric gauge theories on the sphere looked new and rather surprising.

To apply SUSY localization, one first chooses a specific Killing spinor ξ_A, $\bar{\xi}_A$. For generic choice there are two special points on S^4, the north and south poles, characterized respectively by $\xi_A=0$ and $\bar{\xi}_A=0$. If the S^4 is defined by

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = \ell^2, \tag{9}$$

then one can put the north pole at $x_0=\ell$ and the south pole at $x_0=-\ell$ using conformal symmetry. The square of the corresponding supersymmetry yields the sum of rotations about the (x_1,x_2) plane and (x_3,x_4) plane with equal coefficients. In particular, near the two poles the supersymmetry is approximately that of topologically (anti-)twisted theory with Omega deformation $\epsilon_1=\epsilon_2=\ell^{-1}$.

The supersymmetric saddle points are given by the constant value a of the scalar in the vector muliplet, and the hypermultiplet fields have to be all zero. The gauge field is also required to be zero at generic points on S^4 up to gauge choice, but it can take point-like instanton or anti-instanton configurations at the north or south poles. The (anti-)instantons give rise to Nekrasov partition functions from each pole. Thus the full partition function takes the form

$$Z = \int da e^{-S_{\mathrm{YM}}(\tau;a)} Z_{1\text{-loop}}(a, m) Z_{\mathrm{Nek}}(q; a, m, \epsilon_1, \epsilon_2) Z_{\mathrm{Nek}}(\bar{q}; a, m, \epsilon_1, \epsilon_2). \tag{10}$$

Here the integral is over a Cartan subalgebra of G, m is the matter mass, and $q = e^{2\pi i \tau}$ becomes the instanton counting parameter in the Nekrasov partition function. The classical action and one-loop determinant are given by

$$e^{-S_{\mathrm{YM}}} = (q\bar{q})^{\frac{1}{2}\mathrm{Tr}(a^2)}, \quad Z_{1\text{-loop}} = \frac{\prod_{\alpha\in\Delta} \Upsilon(ia\cdot\alpha)}{\prod_{w\in R} \Upsilon(1 + ia\cdot w + im)}, \tag{11}$$

where α runs over the root of G and w is the weight of the representation R. The function $\Upsilon(x)$ here is defined as an infinite product,

$$\Upsilon(x) = (\mathrm{const}) \cdot \prod_{n\geq 1} (x - 1 + n)^n (1 - x + n)^n. \tag{12}$$

As reviewed in the introduction, the one-loop determinant can be evaluated by choosing a suitable $\mathbf{Q}$-exact deformation of the action $\mathbf{Q}\mathcal{V}$, approximating it by a quadratic functional in fluctuations, and evaluating the Gaussian integral. However, the standard choice of $\mathbf{Q}\mathcal{V}$ for this problem does not lead to quadratic functionals which respect $SO(5)$ rotation invariance of S^4, so the direct evaluation of the determinant is very complicated. An elegant solution is to translate the problem into that of the index of (transversally elliptic) differential operators, which essentially evaluates the trace of $e^{-it\mathbf{Q}^2}$ on some reduced Hilbert spaces. If one uses this idea, there is actually no need to explicitly work out the spectrum of any Laplace or Dirac operators. A detailed explanation of how to compute the indices for transversally elliptic differential operators was given in [6], including subtle issues of regularizations. Though mathematically quite involved, the use of the index theorem has become essential in studying SUSY gauge theories, especially in higher dimensions.

One of the main purposes in solving the SUSY gauge theories on S^4 was to give an analytic proof of the conjecture [7,8] that circular Wilson loops in $\mathcal{N} = 4$ super Yang–Mills theory are given by a Gaussian matrix integral. To show this, one chooses the hypermultiplet to be in the adjoint representation of G and apply the result of localization to the so-called $\mathcal{N} = 2^*$ theory. When the mass for the hypermultiplet is turned off, then the one-loop determinant becomes nothing but the Vandermonde determinant. The Nekrasov partition function also becomes trivial: $Z_{\mathrm{Nek}} = 1$. Thus one can explicitly see that the path integral reduces to just the Gaussian matrix integral over a.

2.2. AGT relation

In 2009 there was a series of breakthroughs in 4D $\mathcal{N} = 2$ supersymmetric gauge theories. Gaiotto proposed the construction of families of superconformal field theories of *class S* based on the picture of multiple M5-branes wrapped on punctured Riemann surfaces [11]. Interestingly, for these class models the marginal gauge couplings can be identified with the complex structure moduli of the Riemann surface wrapped by the M5-branes. This led to a geometric interpretation of the strong–weak coupling dualities in gauge theories.

A little later, Alday, Gaiotto, and Tachikawa found a surprising correspondence between a family of gauge theories of class S and the two-dimensional Liouville conformal field theory (CFT) [12]. They studied the theories describing two M5-branes wrapped on a Riemann surface Σ with n punctures. The S^4 partition function and the Nekrasov partition function of the resulting theory T_Σ were then compared with the n-point correlation function of Liouville theory on Σ and its holomorphic building blocks called conformal blocks, and they were shown to agree precisely. Similar correspondence was found between class-S theories of higher rank and Toda conformal field theories by [13]. See the review in [15] for more detail on this correspondence.

Toda theories and the 6D theories on multiple M5-branes both obey ADE classification. The theory on two M5-branes and Liouville theory are both labeled by A_1, the simplest entry in this classification. Let us summarize here the essential facts in Liouville theory and then try to describe how an expert in Liouville theory would have understood the AGT relation when it was first proposed.

2.3. Liouville theory revisited

Liouville theory is a theory of a massless real scalar field ϕ with exponential potential $e^{2b\phi}$, where b is called the Liouville coupling. Though interacting, it is known to be a conformal field theory of central charge

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}. \tag{13}$$

Another remarkable feature of Liouville theory is the self-duality: the theories with couplings b and $1/b$ are known to be equivalent. Thanks to conformal symmetry, correlation functions of arbitrary sets of local operators on general Riemann surfaces can in principle be constructed algebraically from the two- and three-point functions of primary operators on the sphere [16]. The three-point function of primary operators $V_\alpha \equiv \text{const} \cdot e^{2\alpha\phi}$ in Liouville theory,

$$\left\langle V_{\alpha_3}(\infty) V_{\alpha_2}(1) V_{\alpha_1}(0) \right\rangle = C^{(3)}_{\alpha_1,\alpha_2,\alpha_3}, \tag{14}$$

was obtained in [17] and [18].

Conformal blocks are the basic building blocks in the construction of correlators. In general, the dependence of correlation functions of 2D CFT on the moduli τ_i of a punctured Riemann surface (the shape of the surface as well as the position of the insertions) is determined by the conformal Ward identity. They consist of a set of holomorphic differential equations in τ_i and the similar set for $\bar{\tau}_i$. Conformal blocks are the solutions to the set of holomorphic differential equations. There are different choices for the basis of conformal blocks corresponding to different channels in which to express correlators. For example, the diagram on the right of Fig. 1, called a Moore–Seiberg graph, expresses the torus three-point function $\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle_{T^2}$ in a particular channel, in which α_a are external Liouville momenta and β_a the momenta along the internal lines. The conformal blocks $\mathcal{F}$ in this channel are functions of α_a, β_a as well as τ_i. The correlation function can then be expressed as

$$\left\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \right\rangle_{T^2} (\tau_i, \bar{\tau}_i) = \int d^3\beta\, C^{(3)}_{\alpha_1,\alpha_2,\beta_1} C^{(3)}_{\beta_1,\beta_2,\beta_3} C^{(3)}_{\beta_2,\beta_3,\alpha_3} \left| \mathcal{F}_{\vec{\alpha},\vec{\beta}}(\tau_i) \right|^2. \tag{15}$$

Under the AGT relation, the conformal blocks $\mathcal{F}$ are identified with the Nekrasov partition function, and the product of $C^{(3)}$ with one-loop determinants. The momenta $\vec{\alpha}$ and $\vec{\beta}$ correspond to masses m and Coulomb branch parameters a. In particular, each internal line in the Moore–Seiberg graph corresponds to an $SU(2)$ vector multiplet. With all these identifications understood, the formula (15) looks like an S^4 partition function [12].

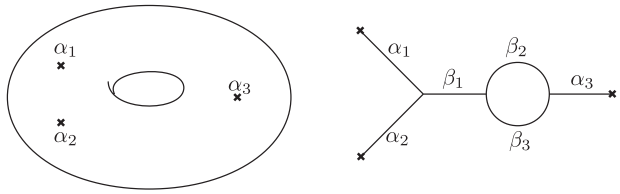

Fig. 1. Torus three-point function and its Moore–Seiberg graph.

Let us now look into the correspondence in more detail. First, it was proposed in [12] that the parameters ϵ_1, ϵ_2 of Omega deformation are related to the Liouville coupling b as

$$\epsilon_1 : \epsilon_2 = b : \frac{1}{b}. \tag{16}$$

This implies that the correspondence between Nekrasov's partition functions and conformal blocks is for general Liouville central charge, but the S^4 partition function should correspond to Liouville correlators at a special (self-dual) value of Liouville coupling $b = 1$, since the Omega background with $\epsilon_1 = \epsilon_2$ showed up near the poles. A natural question, as was already raised in [12], would have been what kind of deformation of S^4 would give the CFT correlators at $b \neq 1$. That led to the idea of *squashing*.

Second, the one-loop determinant $Z_{1\text{-loop}}$ in the S^4 partition function was identified with the product of the Liouville three-point function $C^{(3)}_{\alpha_1,\alpha_2,\alpha_3}$. The analytic property of $C^{(3)}$ can be determined from the following physical requirements of Liouville theory:

- $C^{(3)}$ is symmetric in its three arguments and invariant under $\alpha_1 \to Q - \alpha_1$.
- $C^{(3)}$ vanishes if one of α_a takes a value for degenerate Virasoro representations, $\alpha = -mb - nb^{-1}$ $(m, n \in \mathbb{Z}_{\geq 0})$.
- $C^{(3)}$ diverges if $\alpha_1 + \alpha_2 + \alpha_3 = Q - mb - nb^{-1}$ $(m, n \in \mathbb{Z}_{\geq 0})$, since in this case the Liouville interaction can screen the violation of momentum conservation.

$C^{(3)}_{\alpha_1,\alpha_2,\alpha_3}$ thus has several groups of poles and zeroes, each group containing an infinite number of elements labelled by two nonnegative integers m, n. These should be somehow related to the eigenvalues of $\mathbf{Q}^2 = \epsilon_1 J_{12} + \epsilon_2 J_{34} + (\cdots)$.

The most interesting would have been the correspondence between conformal blocks and Nekrasov's partition functions. In the traditional approach to CFT following [16], the only way to construct and study conformal blocks was via power series in 2D coordinates, or in other words summing up all the descendant operators appearing in the given operator product. There is actually a powerful recursion relation due to Zamolodchikov [19] that can determine the coefficients of higher terms in the series expansion from the lower ones, and it was used in proving the AGT conjecture for some basic examples [20,21]. A better understanding of conformal blocks beyond their definition as power series was definitely needed. This was a rather unexplored subject, although Liouville theory has a long history and has played such an important role in many places in string theory.

Liouville conformal blocks were studied from a different perspective in a series of works by Teschner [22–25]. As we have seen, conformal blocks form a complete basis of solutions to the conformal Ward identity in a given channel. One can therefore study the conformal blocks through their transformation property under changes of basis: namely how the bases of conformal blocks in different channels are related. Under the AGT relation, different channel descriptions of the same correlator are in correspondence with different Lagrangian descriptions of the same 4D quantum field

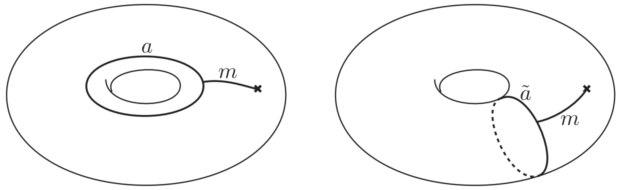

Fig. 2. Two channels for torus one-point conformal blocks.

theory, that is the S-duality. On the other hand, it was known that the Liouville conformal blocks obey the same transformation rule under the change of basis as the wave functions in quantum Teichmüller theory, which is also related to quantization of the moduli space of flat $SL(2,\mathbb{R})$ gauge fields on a punctured Riemann surface. In [26,27] this fact was used as a key to explain how the 4D gauge theories and Liouville theory are related.

For later use, let us look at an example of basis-change of Liouville conformal blocks for a one-point function on the torus. The corresponding Moore–Seiberg graph is a tadpole, and the conformal blocks are functions of the modulus τ of the torus as well the external and internal momenta $\alpha \equiv \frac{Q}{2} + im$, $\beta \equiv \frac{Q}{2} + ia$; see Fig. 2. They transform under the modular S-transformation $\tau \to -1/\tau$ as follows:

$$\mathcal{F}_{m,a}(\tau) = \int d\tilde{a} \sinh(2\pi b\tilde{a}) \sinh(2\pi \tilde{a}/b) \cdot S(a,\tilde{a},m)\mathcal{F}_{m,\tilde{a}}(-1/\tau). \tag{17}$$

Here we chose a different normalization of conformal blocks compared to (15). The integral kernel $S(a,\tilde{a},m)$ is known to take the following form [24]:

$$\begin{aligned} S(a,\tilde{a},m) = 2^{\frac{3}{2}} s_b(-m) \int_{\mathbb{R}} d\sigma\, s_b\left(\sigma + \tilde{a} + \frac{m}{2} + \frac{iQ}{4}\right) s_b\left(-\sigma + \tilde{a} + \frac{m}{2} + \frac{iQ}{4}\right) \\ \cdot s_b\left(\sigma - \tilde{a} + \frac{m}{2} + \frac{iQ}{4}\right) s_b\left(-\sigma - \tilde{a} + \frac{m}{2} + \frac{iQ}{4}\right) \cdot e^{4\pi i a\sigma}, \end{aligned} \tag{18}$$

where $s_b(x)$ is the double sine function

$$s_b(x) = \prod_{m,n\in\mathbb{Z}_{\geq 0}} \frac{\frac{Q}{2} + mb + nb^{-1} - ix}{\frac{Q}{2} + mb + nb^{-1} + ix}. \tag{19}$$

3. 3D $\mathcal{N} = 2$ gauge theories

The idea of SUSY localization was applied to 3D supersymmetric Chern–Simons matter systems by Kapustin, Willett, and Yaakov (KWY) [9]. Chern–Simons matter theories are a canonical example of 3D superconformal field theories (SCFTs), and some of them are known to have interpretations as theories of multiple M2-branes. Indeed, the original motivation of KWY was to provide a precise check of AdS/CFT through the explicit evaluation of Wilson loops. Moreover, their formula was also applied to, and gave an elegant solution of, the long-standing problem of the growth $\sim N^{3/2}$ of the degree of freedom on multiple M2-branes. Their result also found applications and generalizations in many other interesting problems, some of which we review in the following.

3.1. S^3 partition function

KWY constructed supersymmetric Chern–Simons matter theories on S^3 and obtained a closed formula for the SUSY partition function as well as Wilson loop expectation values, which apply to a class of 3D $\mathcal{N} = 2$ supersymmetric systems. The system consists of two kinds of multiplets: a vector multiplet consists of a gauge field A_m, gauginos λ_α, $\bar{\lambda}_\alpha$, real scalar σ, and an auxiliary scalar D. A chiral multiplet consists of a complex scalar ϕ, fermion ψ, and a complex auxiliary scalar F, and can couple to a vector multiplet in an arbitrary representation R of the gauge group. The gauge fields have Chern–Simons kinetic term

$$S = \frac{k}{4\pi} \int \mathrm{Tr}\left(\mathrm{Ad}A + \frac{2}{3}A^3\right), \tag{20}$$

where k is the quantized Chern–Simons coupling. For each $U(1)$ factor of the gauge group one can also turn on the Fayet–Iliopoulos coupling ζ. For chiral multiplets, in addition to standard gauge interactions one can turn on other interactions through superpotential, or turn on the so-called real mass through gauging global symmetry. The supersymmetric Lagrangian and transformation rules can be written down based on the existence of conformal Killing spinors on S^3,

$$D_\mu \epsilon \equiv \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma^{ab}\right)\epsilon = \gamma_\mu \tilde{\epsilon} \text{ for some } \tilde{\epsilon}. \tag{21}$$

An important restriction, to which we will come back later, is that all the chiral multiplets here are assigned canonical R-charge 1/2.

The exact S^3 partition function depends on G_k (convenient notation for the gauge group and its Chern–Simons coupling), and chiral matter representation R. The formula reads

$$Z = \int d^r\sigma e^{i\pi k \mathrm{Tr}(\sigma^2)} \prod_{\alpha\in\Delta_+} (2\sinh \pi\alpha\cdot\sigma)^2 \prod_{w\in R} F(w\cdot\sigma), \tag{22}$$

where

$$F(x) \equiv \prod_{n\geq 1}\left(\frac{n+\frac{1}{2}+ix}{n-\frac{1}{2}-ix}\right)^n = s_{b=1}\left(\frac{i}{2} - x\right). \tag{23}$$

The FI coupling ζ shows up as a modification of the integrand by $e^{4\pi i\zeta\sigma}$ [28].

With SUSY localization, the path integral can be shown to simplify to a finite-dimensional integral over constant values of vector multiplet scalar σ, which one can further restrict to the Cartan subalgebra. An important simplification compared to the four-dimensional case is the absence of saddle points with non-trivial topological quantum numbers such as instantons. Another simplification is that the one-loop determinant here can be evaluated explicitly as a product of eigenvalues using spherical harmonics, and the evaluation essentially boils down to representation theory of $SU(2)$. Their formula is thus very easy to reproduce, so in a sense the 3D theories on S^3 can be thought of as an ideal exercise to learn the essence of SUSY localization.

3.2. *Application to M2-brane theories*

An important application of the KWY formula is to the multiple M2-brane dynamics and AdS_4/CFT$_3$ correspondence. In this area, a long-standing problem was how to understand the growth of the degrees of freedom (or free energy) on N coincident M2-branes $\sim N^{3/2}$ predicted by the dual supergravity description. If the worldvolume theory on a stack of N M2-branes is described by a 3D gauge theory with $N \times N$ matrix-valued fields, then the naive count of the degrees of freedom would be $\sim N^2$. The description of the multiple M2-branes worldvolume theory itself was a long-standing problem, but in [29] an $\mathcal{N} = 6$ superconformal $U(N)_k \times U(N)_{-k}$ Chern–Simons theory

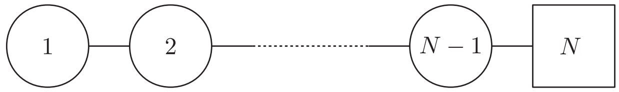

Fig. 3. Quiver diagram for the theory $T[SU(N)]$.

with bi-fundamental matters was proposed for N M2-branes on orbifold $\mathbb{R}^8/\mathbb{Z}_k$. Indeed, it is a theory of $N \times N$ matrix-valued fields, while the dual supergravity predicts the large N behavior for the free energy

$$F \sim \frac{\sqrt{2}\pi}{3} k^{1/2} N^{3/2}. \tag{24}$$

An elegant solution for this mismatch was proposed in [30] by applying the traditional methods of large-N matrix integrals to the S^3 partition function of the ABJM model. They in particular found that the standard't Hooft expansion of the logarithm of the sphere partition function reproduces (24) in its leading order. The subleading contributions as well as instanton contributions were studied in detail using various approaches to evaluate the integral (22), and interpreted in the dual picture. See the review in [31] for more detail. Note that the fact that the S^3 partition function admits such an expansion or resummation is important in view of the AdS/CFT correspondence. The observables in the gauge theory side need to have a well-defined analytic continuation in N, because N is mapped to the cosmological constant on the gravity side.

3.3. *The AGT relation in 3D*

In 4D gauge theories, one can introduce various defects and study them. According to their dimensionality they are called loops, surface defects, or domain walls (or boundaries). It is especially interesting to study how to describe them using lower-dimensional field theories, or how the duality in 4D gauge theories act on them. Certain domain walls in 4D $\mathcal{N} = 2$ supersymmetric gauge theories are described by 3D $\mathcal{N} = 2$ field theories, and the sphere partition function gives important information on their properties.

The study of domain walls and boundaries for this purpose was started in $\mathcal{N} = 4$ SYM by Gaiotto and Witten [32,33]. They were particularly interested in how the Montonen–Olive $SL(2, \mathbb{Z})$ duality of the SYM acts on the boundaries and domain walls. As an example, consider the SYM with gauge group G and take a half-BPS completion of the Dirichlet boundary condition on the gauge field. Its S-dual was then shown to be a 3D $\mathcal{N} = 4$ SCFT called $T[G]$ on the boundary coupled to the bulk SYM with the S-dual gauge group ${}^L G$. The theory $T[G]$ is characterized by its global symmetry $G \times G^L$ where G^L is the gauge group for the S-dual theory. For $G = SU(N)$, the wall theory has the 3D $\mathcal{N} = 4$ quiver description as in Fig. 3.

For example, $T[SU(2)]$ is the $U(1)$ SQED with two charged hypermultiplets. A copy of $SU(2)$ acts as flavor rotation, while another $SU(2)$ isometry shows up as the isometry of the Coulomb branch moduli space $\mathbb{C}^2/\mathbb{Z}_2$ in the infrared. The theory $T[G]$ can also be used to describe the S-duality domain wall, that is the interface where the two $\mathcal{N} = 4$ SYM theories with gauge groups G and G^L are adjoined.

The structure of S-duality should be even richer for $\mathcal{N} = 2$ supersymmetric theories. As reviewed in the previous section, two mutually S-dual theories are related in the same way as the conformal blocks in two different channels are related. Then what kind of 3D theory shows up at the joint of a pair of mutually S-dual theories? Though general constructions of such theories were not available, it was conjectured in [34] that the S^3 partition function of the theory on the wall should correspond to the transformation coefficients of conformal blocks under changes of channels, such as the example (18).

An attempt to see the correspondence was made in [35], which studied the S-duality wall between two 4D half-spaces both supporting $\mathcal{N} = 2^*$ theory with $G = SU(2)$. The fields on the two sides are connected across the wall via S-duality. The vacua on the two sides are specified by two Coulomb branch parameters $a, \tilde{a}$. The theory on the wall was identified as a suitable mass deformation of the theory $T[SU(2)]$ explained above. In 3D $\mathcal{N} = 2$ terminology, it consists of a $U(1)$ vector multiplet, two chiral multiplets q_1, q_2 of charge $+1$, two chirals $\tilde{q}^1, \tilde{q}^2$ of charge -1, and a neutral chiral ϕ. The chiral matters all acquire mass proportional to the bulk $\mathcal{N} = 2^*$ mass deformation m. In addition, the parameters $a, \tilde{a}$ enter the theory as the FI parameter and the mass for charged chirals.

When computing the S^3 partition function for the wall theory, a small but nontrivial problem arose. The neutral chiral multiplet ϕ of the wall theory is assigned the R-charge 1, for which the one-loop determinant was not derived. Without knowing the contribution from ϕ it was proposed in the first version of [35] that

$$Z_{S^3}(a, \tilde{a}, m) = \text{const} \cdot \int d\sigma\, s_{b=1}\left(\sigma + \tilde{a} + \frac{m}{2} + \frac{i}{2}\right) s_{b=1}\left(-\sigma + \tilde{a} + \frac{m}{2} + \frac{i}{2}\right) \cdot s_{b=1}\left(\sigma - \tilde{a} + \frac{m}{2} + \frac{i}{2}\right) s_{b=1}\left(-\sigma - \tilde{a} + \frac{m}{2} + \frac{i}{2}\right) \cdot e^{4\pi i a\sigma}. \tag{25}$$

Though the analysis was incomplete, this result shows quite an agreement with (18). Thus it was proposed that the AGT relation can be generalized to include domain walls, and there is a precise relation between 3D gauge theories and 2D CFTs.

The above observation of the correspondence between 3D gauge theories and 2D CFTs was soon generalized in an interesting manner. To explain it, let us recall that the S-duality domain walls are closely related to Janus domain walls connecting the same 4D gauge theories at different values of coupling. As a generalization of the Janus wall, let us consider the situation in which the gauge coupling varies smoothly as a function of one of the spatial coordinates, say x_3. For theories of class S, the situation corresponds to M5-branes wrapping some Riemann surface whose shape varies as a function of x_3. One can reinterpret it as M5-branes wrapping a 3-manifold. This picture leads to a correspondence between the geometry of hyperbolic 3-manifolds $\mathcal{M}$ and the corresponding 3D $\mathcal{N} = 2$ gauge theories $T[\mathcal{M}]$, as proposed in [36]. Moreover, a correspondence which is similar to the AGT relation was proposed between observables of $T[\mathcal{M}]$ and Chern–Simons path integrals on $\mathcal{M}$ [36–39], and various precise correspondences have been reported.

3.4. Generalization of KWY formula

On a closer look at the formulae (18) and (25), it is tempting to identify $s_b(-m)$ with the one-loop determinant of the neutral chiral multiplet, as was proposed in the second version of [35]. It is also tempting to look for deformations of the round S^3 which reproduces the double sine function for general b, as we will discuss in the next section.

Finding the one-loop determinant arising from the ϕ of non-canonical R-charge assignment requires generalizing the construction of supersymmetric theories on S^3 accordingly. This turned out to be possible, and what is intriguing was that the supersymmetry transformation rule for chiral multiplet (ϕ, ψ, F) then depends explicitly on its R-charge q:

$$
\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi, \\
\delta\psi &= i\gamma^\mu\epsilon D_\mu\phi + \frac{2qi}{3}\gamma^\mu D_\mu\epsilon\phi + \bar{\epsilon}F, \\
\delta F &= i\epsilon\gamma^\mu D_\mu\psi + \frac{i}{3}(2q-1)D_\mu\epsilon\gamma^\mu\psi.
\end{aligned}
\tag{26}
$$

Similar R-charge dependence also shows up in the Lagrangian. The SUSY localization computation of the S^3 partition function goes through, and the one-loop determinant for the chiral multiplet of R-charge q was found to be

$$
F_q(x) \equiv \prod_{n\geq 1}\left(\frac{n+1-q+ix}{n-1+q-ix}\right)^n = s_{b=1}(i-iq-x), \tag{27}
$$

generalizing (23). One can check using this formula that the neutral chiral multiplet of mass $-m$, R-charge 1 gives rise to the determinant $s_b(-m)$, which completes the agreement. This was reported by Jafferis [40], and one day later by [41].

3.5. *F-theorem*

Thanks to the above generalization, arbitrary $\mathcal{N}=2$ supersymmetric theories with R-symmetry can now be put on S^3 preserving rigid supersymmetry. For theories with Abelian global symmetry, the assignment of R-charges to chiral matters is not unique; any two consistent assignments, $q_i = R[\phi_i]$ and $q_i' = R'[\phi_i]$, differ by a linear combination of Abelian global symmetry charges $Q_a[\phi_i]$. Given a reference R-charge R_0, one can parametrize different assignments of R-charges in the following way,

$$
R = R_0 + \sum_a t_a Q_a. \tag{28}
$$

The S^3 partition function then becomes a function of the parameters t_a.

If the theory flows to a superconformal field theory in the infrared, then the R-symmetry in the IR limit is uniquely defined as a member of the superconformal algebra. Jafferis [40] made an interesting proposal that the corresponding value of t_a can be determined by extremizing the real part of the free energy $F_{S^3}(t) = -\log Z_{S^3}(t)$. This was proved in [42] based on a careful study of the structure of couplings between current supermultiplets of the field theory and the background supergravity multiplet.

4. Squashing

The comparison of the formulae (18) and (25) leads to another natural guess that the S^3 partition function should be deformed in some way to reproduce the quantities in Liouville theory with $b \neq 1$. We encountered the same unsatisfactory situation also in the comparison of 4D and 2D observables, but the deformation of S^3 gauge theories seems easier to find.

It turned out that the rigid supersymmetry can be realized on manifolds less symmetric than the round sphere, and moreover one can derive exact formulae for supersymmetric observables on such manifolds. The important examples are squashed spheres. It was shown that for a suitable deformation of S^3 the formula (22) is modified to exhibit the expected b dependence.

4.1. Ellipsoid partition function

One way to achieve $b \neq 1$ is to deform the round sphere into an ellipsoid [43],

$$\frac{1}{\ell^2}\left(x_1^2 + x_2^2\right) + \frac{1}{\tilde{\ell}^2}\left(x_3^2 + x_4^2\right) = 1, \tag{29}$$

and generalize the Killing spinor equation to include a background $U(1)_{\mathrm{R}}$ gauge field V_μ,

$$\begin{aligned} D_\mu \epsilon &= \left(\partial_\mu - iV_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma^{ab}\right)\epsilon = \frac{iH}{2}\gamma_\mu \epsilon, \\ D_\mu \bar{\epsilon} &= \left(\partial_\mu + iV_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma^{ab}\right)\bar{\epsilon} = \frac{iH}{2}\gamma_\mu \bar{\epsilon}. \end{aligned} \tag{30}$$

The scalar function H and the gauge field V_μ are suitably chosen so that the above equations have solutions. Then the SUSY localization leads to the following formula for the partition function:

$$Z = \int d^r\sigma\, e^{i\pi k \mathrm{Tr}(\sigma^2)} \prod_{\alpha\in\Delta_+} 4\sinh(\pi b\alpha\sigma)\sinh\left(\pi b^{-1}\alpha\sigma\right) \cdot \prod_{w\in R} s_b\left(\frac{iQ}{2}(1-q) - w\sigma\right), \tag{31}$$

which generalizes (22). The Liouville coupling b was shown to be related to the axis lengths by $b = (\ell/\tilde{\ell})^{1/2}$.

4.2. Sketch of derivation

The idea of ellipsoidal deformation naturally comes about from the following observation. In Pestun's derivation of the S^4 partition function, one-loop determinants were evaluated by relating them to the determinant of the bosonic symmetry $\mathbf{Q}^2$ on some reduced space of wavefunctions. It is reasonable to expect that $\mathbf{Q}^2$ plays a similar role in three dimensions as well. On the round sphere $x_0^2 + x_1^2 + x_2^2 + x_3^2 = \ell^2$, the localization analysis was based on the Killing spinor $\epsilon, \bar{\epsilon}$ satisfying

$$\mathbf{Q}^2 = i\bar{\epsilon}\gamma^m\epsilon\partial_m + \cdots = \frac{i}{\ell}(J_{12} + J_{34}) + \cdots,$$

where J_{12}, J_{34} are the generators of rotations of $\mathbb{R}^4$. This choice of Killing spinor is essentially unique due to the isometry of S^3. Then a natural guess is that, if there were deformations of the sphere for $b \neq 1$, the square of the corresponding SUSY should be deformed in the following way:

$$\mathbf{Q}^2 = i(J_{12} + J_{34}) + \cdots \quad \longrightarrow \quad \mathbf{Q}^2 = ib^{-1}J_{12} + ibJ_{34} + \cdots. \tag{32}$$

The deformed geometry therefore should be $U(1) \times U(1)$ symmetric, and ellipsoids (29) with the identification $\ell : \tilde{\ell} = b : b^{-1}$ is an natural guess. However, at this level the idea is still too crude, because the conformal Killing spinor equation (30) was known to have solutions only on a rather restricted class of manifolds. Indeed, one can try to solve (30) with various $U(1) \times U(1)$ symmetric ansätze for the metric and see that none of the attempts work except for the round sphere.

During the process of trial and error, we got interested in how the Killing spinor equation on the round S^3 would break down by small deformations of the metric while keeping the Killing spinor unchanged. Since the problem is to find a family of geometries parametrized by b, one can work perturbatively near $b = 1$. If the small deformation to the geometry were suitably chosen, we could fix the failure of the Killing spinor equation somehow by modifying the Killing spinor accordingly. As the first experiment, the deformation of the round sphere into what was traditionally called a squashed sphere was considered:

$$ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2 + \mu^3\mu^3) \longrightarrow ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2) + \tilde{\ell}^2\mu^3\mu^3. \tag{33}$$

Here $\mu^a = \mu^a_\mu dx^\mu$ is the basis of left-invariant one-forms of $SU(2)$. For a suitable choice of Killing spinor ϵ on the round sphere, the failure after deformation turned out to be

$$\left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma^{ab}\right)\epsilon - \frac{i\tilde{\ell}}{2\ell^2}\gamma_\mu\epsilon = \pm i\left(1 - \frac{\tilde{\ell}^2}{\ell^2}\right)\mu^3_\mu\epsilon. \tag{34}$$

The original plan was to modify ϵ so that the failure term (RHS) disappears, but the above equation seemed to suggest a much nicer alternative solution. One can just regard the failure term as a coupling to a background vector field V_μ and include it in the covariant derivative.

It is a tedious but pleasant exercise to check that the construction of the SUSY transformation rule and Lagrangians all goes through, under the assumption that $\epsilon, \bar{\epsilon}$ are assigned the V_μ-charge ± 1. In particular, all the fields can be shown to couple to V_μ according to their R-charge q in (26), thus V_μ can really be identified as the gauge field for $U(1)$ R-symmetry. At this point, however, we were not sure what kind of framework would naturally accommodate this external gauge field. The external gauging of R-symmetry was regarded just as a tool to define supersymmetry on curved space, in a similar way to topological twisting.

There was no particular reason to consider the deformation to a traditional *squashed sphere* (33), but in this way one is left with a large isometry unbroken. The spectrum on this space can therefore be explicitly solved using spherical harmonics. In some old literature there are even explicit results on related problems [44]. After a detailed spectrum analysis, we found that the eigenfunctions can be written using spherical harmonics in the same way as for the round sphere, but the degeneracy of eigenvalues gets partially resolved due to squashing. We were hoping that this broken degeneracy would lead to something new. But, disappointingly, the one-loop determinants stayed essentially the same as those for the round sphere.

After all, the square of supersymmetry on the traditional squashed sphere does not show the expected dependence on b (32). Also, the eigenmodes turned out to make nontrivial contributions to the one-loop determinant as multiplets of the unbroken $SU(2)$ isometry, so that the determinant still has a degeneration of many zeroes and poles. Thus it looked inevitable to break the isometry further and try seriously the ellipsoid (29).

Coming back from disappointment, it was pleasing to see that the ellipsoid (29) also admits charged Killing spinors if a suitable background $U(1)_\mathrm{R}$ gauge field V_μ is turned on. Moreover, this time the bilinear of the Killing spinors indeed showed the expected b dependence (32). The only remaining problem was how to compute one-loop determinants.

On the ellipsoid (29) there seemed to be no easy way to solve the full spectrum. On the other hand, it was clear from previous experiences of determinant computations that most eigenmodes form Bose–Fermi pairs and do not make nontrivial contributions. It is therefore enough to know the spectrum of the remaining "unpaired modes." It seemed difficult to translate our problem completely mathematically into the computation of an index as in [6]. Instead, in [43] the problem was studied in an equivalent and a little more physical approach by asking the following questions:

- What are the Laplace and Dirac operators one wishes to know the eigenvalues of?
- What is the map between the Laplace and Dirac eigenmodes for the same eigenvalue?

The one-loop determinant can then be expressed by collecting the eigenvalues of those unpaired modes which are sitting in the kernel and cokernel of the map. It turned out that all the unpaired modes can be easily listed as solutions to some simple first-order differential equations. We thus arrived at an analytic result (31) which shows precisely the expected dependence on a new parameter $b = (\ell/\tilde{\ell})^{1/2}$.

The analysis of one-loop determinants on the ellipsoid was revisited later and translated into the computation of indices in [45,46].

4.3. Relation to superconformal index

It was noticed in [47,48] that the 3D partition functions for $\mathcal{N} = 2$ theories have structures similar to superconformal indices for 4D $\mathcal{N} = 1$ theories—see [49] for a review. The superconformal index is an observable which encodes the spectrum of BPS operators, and is usually defined as the trace of time evolution operators over Hilbert space with an additional insertion of $(-1)^F$. Alternatively, one can use the path integral formulation and define it as a partition function on $S^1 \times S^3$ with SUSY-preserving periodicity condition on fields. The relations between 3D partition functions and 4D indices were studied from this viewpoint in [50].

One can introduce a one-parameter deformation to the 4D superconformal index which is similar to the squashing of the 3D partition function by twisting the periodicity of fields along S^1 by isometry rotation of S^3. Interestingly, if the 4D theory with this twist is dimensionally reduced, the resulting 3D theory is actually on the traditional squashed sphere (33), somewhat against our previous observation which led to the ellipsoid partition function. This led Imamura and Yokoyama to find another supersymmetric deformation of the round S^3 by introducing a background vector field [51].

4.4. Further generalization and supergravity

It is natural to ask what other three-manifolds admit rigid $\mathcal{N} = 2$ supersymmetry, and what is the maximum consistent generalization of the Killing spinor equation. Festuccia and Seiberg [52] proposed that the most suitable framework for such a study is off-shell supergravity. The background fields introduced in (30) or in [51] are then most naturally interpreted as the (auxiliary) fields in the gravity multiplet, and the Killing spinor equation is identified with the vanishing of the local SUSY transformation of the gravitino. Regarding the existence of rigid supersymmetry on curved space, it was shown that a 3D space admits a Killing spinor if it has an almost contact metric structure [53,54]. The general theory of how the 3D partition function can depend on moduli of almost contact metric structure (such as the squashing parameter b) was developed in [55]. In particular, it was shown that a partition function on three-manifolds of the topology of S^3 cannot depend on more than one squashing parameters [56].

4.5. Squashing S^4

After an instructive detour to three dimensions, we finally came back to the problem of finding a deformation of S^4 which reproduces Liouville correlators with $b \neq 1$. A natural answer was proposed in [14] based on the 4D ellipsoid geometry

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1, \tag{35}$$

with some auxiliary fields in 4D $\mathcal{N} = 2$ off-shell supergravity turned on. Let us now sketch how this result was derived.

The first step was to identify the correct generalization of the Killing spinor equation (7), and then use it to construct the transformation rule and Lagrangian. This analysis was started before the observation of Festuccia and Seiberg [52], so the usefulness of supergravity was not yet recognized. So the only idea to generalize the Killing spinor equation (7) was to turn on R-symmetry gauge fields. Concerning the metric on the 4D manifold that realizes $b \neq 1$, it seemed natural to assume a fibration structure in which a 3D ellipsoid is fibered over a segment, with the fiber size shrinking at the two ends. The ellipsoid (35) is clearly one such example, where $x_0 \in [-r, r]$ is the coordinate on the base segment and a 3D ellipsoid of varying size is fibered over it.

It was contrary to our optimistic expectations and even surprising that the ellipsoid does not admit Killing spinors no matter how one chooses the R-symmetry gauge field. After a more systematic study of the ellipsoid–fibration geometries, one 4D metric was found to admit Killing spinors, but it turned out to have a rather strange singularity at the two poles (points at the end of the segment). It seemed somewhat awkward to discuss the physics of point-like instantons localized on such a singular point.

The first nontrivial step was made by recalling that near the north pole our Killing spinor of interest should represent the SUSY of a topologically twisted theory on the Omega background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$. There the chiral part of Killing spinor ξ_A vanishes while the anti-chiral part $\bar{\xi}_A$ is finite. By a suitable gauge rotation one may set $\bar{\xi}^{\dot{\alpha}}_A = \text{const} \cdot \delta^{\dot{\alpha}}_A$ at the north pole, since in topologically twisted theory one identifies dotted spinor indices and $SU(2)$ R-symmetry indices. We also need that the square of the SUSY give rise to a rotation about the origin generated by the vector field

$$v^m \equiv 2\,\bar{\xi}^A \bar{\sigma}^m \xi_A = (-\epsilon_1 x_2,\ +\epsilon_1 x_1,\ -\epsilon_2 x_4,\ +\epsilon_2 x_3), \tag{36}$$

where x_i are local Cartesian coordinates near the north pole. This determines the linear dependence of ξ_A on coordinates

$$\xi_A = \tfrac{1}{2} v^m \sigma_m \bar{\xi}_A. \tag{37}$$

Now let us perform the failure term analysis in a similar way to the 3D case. On a flat $\mathbb{R}^4$ without background gauge fields, the Killing spinor $\xi_A, \bar{\xi}_A$ satisfies

$$D_m \bar{\xi}_A = 0, \quad D_m \xi_A + \tfrac{1}{8} v^-_{kl} \sigma_{kl} \cdot \sigma_m \bar{\xi}_A = \sigma_m \cdot \left(\tfrac{1}{8} v^+_{kl} \bar{\sigma}_{kl} \bar{\xi}_A \right). \tag{38}$$

Here $v_{kl} = \partial_k v_l = \partial_{[k} v_{l]}$, and the suffix $\pm$ indicates the self-dual or anti-self-dual components of two-forms. The failure term is in the second equation, the second term in the left-hand side. The tensor v^-_{kl} has nonvanishing components $v^-_{12} = -v^-_{34} = \frac{1}{2}(\epsilon_2 - \epsilon_1)$, and it vanishes near the north pole if the squashing deformation is turned off, $\epsilon_1 = \epsilon_2 = 1/\ell$. The above failure term seemed to suggest a rather unexpected form of generalized Killing spinor equation:

$$\begin{aligned} D_m \xi_A + T^{kl} \sigma_{kl} \sigma_m \bar{\xi}_A &= -i \sigma_m \bar{\xi}'_A, \\ D_m \bar{\xi}_A + \bar{T}^{kl} \bar{\sigma}_{kl} \bar{\sigma}_m \xi_A &= -i \bar{\sigma}_m \xi'_A \quad \text{for some } \xi'_A, \bar{\xi}'_A, \end{aligned} \tag{39}$$

which involves anti-self-dual tensor T^{kl} and self-dual tensor $\bar{T}^{kl}$ auxiliary fields in addition to the R-symmetry gauge fields in D_m.

It was an enjoyable, though tedious, exercise to construct the transformation rule and Lagrangian based on the above generalized Killing spinor equation. One complication was that one needs to require another set of equations on the Killing spinor,

$$\begin{aligned} \sigma^m \bar{\sigma}^n D_m D_n \xi_A + 4 D_l T_{mn} \sigma^{mn} \sigma^l \bar{\xi}_A &= M \xi_A, \\ \bar{\sigma}^m \sigma^n D_m D_n \bar{\xi}_A + 4 D_l \bar{T}_{mn} \bar{\sigma}^{mn} \bar{\sigma}^l \xi_A &= M \bar{\xi}_A, \end{aligned} \tag{40}$$

with M another auxiliary field. This looked strange, since this kind of equation involving the square of a Dirac operator is usually automatically satisfied under the assumption of the first-order equations (39).

The proposal of Festuccia and Seiberg came out a little later. The generalized form (39) of Killing spinor equation turned out entirely consistent with the off-shell $\mathcal{N}=2$ supergravity literature [57,58], and the fields T^{kl}, $\bar{T}^{kl}$, M were identified with the auxiliary fields in the gravity multiplet. Also, the additional Killing spinor equation (40) was identified with the local SUSY transformation rule of an auxiliary spin-1/2 fermion in the gravity multiplet, thereby explaining why the two sets of equations (39) and (40) are independent.

The toughest part of the analysis was to show that the ellipsoid (35) indeed has a Killing spinor for a suitable choice of background auxiliary fields. The strategy of [14] was to assume that a suitably chosen Killing spinor on the round S^4 remains a solution to the Killing spinor equation after squashing. This requirement turns the Killing spinor equation into a set of algebraic equations on the background supergravity fields. They looked highly overdetermined, but turned out to have a family of solutions which depends on three arbitrary functions invariant under $U(1)\times U(1)$ isometry. See [14] for the explicit form of the auxiliary fields. Thus the ellipsoid (35) was finally shown to admit rigid supersymmetry.

4.6. SUSY localization on S_b^4

The SUSY localization analysis on the ellipsoid [14] begins by arguing, based on the continuity in the squashing parameter b, that the SUSY saddle points are parametrized by a constant a in the same way as on the round S^4. Strictly speaking this assumption should be verified. For the case of 3D squashing the saddle-point analysis was carefully fully performed in [56]. Anyway, once this point is settled, the rest of the analysis is a straightforward application of the localization program.

Again, at all the saddle points the gauge fields have to vanish on generic points on the ellipsoid, but it is allowed to have point-like instanton or anti-instanton configurations at the two poles. Moreover, near the poles the theory approaches the topologically twisted theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ with two independent Omega-deformation parameters $\epsilon_1=\ell^{-1}$, $\epsilon_2=\tilde{\ell}^{-1}$.

Let us finally quickly summarize the essence of one-loop determinant computation and how it can be reduced to the computation of an index. The computation of a one-loop determinant involves a Gaussian integral over all the fluctuation modes at a give saddle point. Generally, one can choose as path integration variables a set of bosonic fields $\mathbf{X}$, a set of fermionic fields $\boldsymbol{\Xi}$, and their superpartners $\mathbf{QX}$, $\mathbf{Q}\boldsymbol{\Xi}$. The supersymmetric measure is then

$$\langle\cdots\rangle=\int[\mathcal{D}\mathbf{X}][\mathcal{D}(\mathbf{QX})][\mathcal{D}\boldsymbol{\Xi}][\mathcal{D}(\mathbf{Q}\boldsymbol{\Xi})]\,(\cdots). \tag{41}$$

The one-loop determinant is evaluated as an integral with any $\mathbf{Q}$-exact Gaussian weight $e^{-\mathbf{Q}\mathcal{V}}$. Let us take

$$\begin{aligned}\mathbf{Q}\mathcal{V}&=\mathbf{Q}\left\{(\mathbf{X},\mathbf{QX})+(\boldsymbol{\Xi},\mathbf{Q}\boldsymbol{\Xi})\right\}\\&=(\mathbf{QX},\mathbf{QX})+(\mathbf{Q}\boldsymbol{\Xi},\mathbf{Q}\boldsymbol{\Xi})+\left(\mathbf{X},\mathbf{Q}^2\mathbf{X}\right)-\left(\boldsymbol{\Xi},\mathbf{Q}^2\boldsymbol{\Xi}\right).\end{aligned} \tag{42}$$

Then the one-loop determinant is simply the square root of the ratio of determinants of $\mathbf{Q}^2$,

$$Z_{1\text{-loop}}=\left(\frac{\mathrm{Det}_{\boldsymbol{\Xi}}(\mathbf{Q}^2)}{\mathrm{Det}_{\mathbf{X}}(\mathbf{Q}^2)}\right)^{1/2}. \tag{43}$$

It is instructive to see how all these work in examples with finite numbers of integration variables. In the toy example of the volume of a sphere (1), the supersymmetry $\mathbf{Q}=d-\epsilon i_v$ acts on the local coordinates $\mathbf{X} = (x, y)$, $\boldsymbol{\Xi} =$ (empty set) near the north pole as

$$x \xrightarrow{\mathbf{Q}} dx \xrightarrow{\mathbf{Q}} \epsilon y, \quad y \xrightarrow{\mathbf{Q}} dy \xrightarrow{\mathbf{Q}} -\epsilon x. \tag{44}$$

The above formula can be used to explain the determinant at the north pole.

Application of this idea to the path integral of supersymmetric field theories involves renaming of fields. For example, the 4D $\mathcal{N} = 2$ vector multiplet consists of 10 bosons and 10 fermions after gauge fixing: the physical fields A_m, ϕ, $\bar{\phi}$, λ_A, $\bar{\lambda}_A$, D_{AB}, ghosts c, $\bar{c}$, and Lautrup–Nakanishi field B. We take $\mathbf{Q}$ as a combination of supersymmetry for a specific choice of Killing spinor ξ_A, $\bar{\xi}_A$ and BRST symmetry, and reorganize these fields under its action. For example, gauge field A_m is a member of the set $\mathbf{X}$, whereas its superpartner

$$\Psi_m \equiv i\xi^A\sigma_m\bar{\lambda}_A - i\bar{\xi}^A\bar{\sigma}_m\lambda_A + D_m c \tag{45}$$

is a member of $\mathbf{QX}$. The 10+10 fields are thus divided into four groups, $\mathbf{X}$, $\mathbf{QX}$, $\boldsymbol{\Xi}$, $\mathbf{Q}\boldsymbol{\Xi}$, each consisting of five fields.

The ratio of the determinant (43) can be further simplified if $\mathbf{Q}^2$ acting on the fields $\mathbf{X}$ and $\boldsymbol{\Xi}$ has common eigenvalues. In particular, if there is a differential operator D which relates the fields $\mathbf{X}$ to $\boldsymbol{\Xi}$ and commutes with $\mathbf{Q}^2$, then the ratio of determinants can be computed from the index

$$\begin{aligned} \mathrm{Ind}(D) &= \mathrm{Tr}_{\mathbf{X}}\left(e^{-i\mathbf{Q}^2 t}\right) - \mathrm{Tr}_{\boldsymbol{\Xi}}\left(e^{-i\mathbf{Q}^2 t}\right) \\ &= \mathrm{Tr}_{\mathrm{Ker}D}\left(e^{-i\mathbf{Q}^2 t}\right) - \mathrm{Tr}_{\mathrm{Coker}D}\left(e^{-i\mathbf{Q}^2 t}\right). \end{aligned} \tag{46}$$

Note that the operator D is in principle arbitrary as long as it commutes with $\mathbf{Q}^2$, and it does not necessarily have to be related to the Lagrangian of the field theory. At this point, a powerful localization theorem in mathematics says the index can be computed as a sum over contributions from $\mathbf{Q}^2$-fixed points, so we need the precise form of D only near the poles. The reason for this localization is that $e^{-i\mathbf{Q}^2 t}$ involves a finite rotation (diffeomorphism). If it acts on coordinates as $x^m \mapsto \tilde{x}^m$, then the trace of such an operator should involve an integral of the delta function,

$$d^4x\delta^4(x - \tilde{x}) = \det(1 - \partial\tilde{x}/\partial x)^{-1}, \tag{47}$$

so it localizes onto fixed points. For more details see [6,14], as well as a review [59].

The one-loop determinant $Z_{1\text{-loop}}$ for $\mathcal{N} = 2$ gauge theories on the ellipsoid was thus shown to take the same form (11), with the following b-dependent modification of the function: $\Upsilon(x)$,

$$\Upsilon(x) = (\text{const}) \cdot \prod_{m,n\geq 0} \left(mb + nb^{-1} + x\right)\left(mb + nb^{-1} + Q - x\right). \tag{48}$$

This function was indeed used to express Liouville three-point functions [17,18].

5. Concluding remarks

Let us briefly mention the progress in other dimensions. In five dimensions, the sphere partition function for supersymmetric gauge theories was studied in [60–68]. There the important problem was to see how the sphere partition functions for maximally supersymmetric SYM are related to the index of 6D (2,0) superconformal theories, and to read off the large-N scaling of the degrees of freedom on N coincident M5-branes $\sim N^3$. In two dimensions, the sphere partition function for $\mathcal{N} = (2, 2)$ gauge theories was studied in [69–71]. In particular, for those which flow to $\mathcal{N} = (2, 2)$

superconformal field theories, it was shown that the sphere partition function computes directly the Kähler potential for the moduli space of superconformal theories.

Localization techniques have been applied to the evaluation of many supersymmmetric observables. In addition to partition functions, various non-local observables such as Wilson loops, 't Hooft loops, and surface operators have also been studied using this technique. They are not only playing important roles in understanding the mathematical structures underlying supersymmetric gauge theories, but also help us to better understand how to define and compute such operators precisely within path integral formalism.

Acknowledgements

The author would like to thank Naofumi Hama, Sungjay Lee, and Jaemo Park for collaboration on the materials reviewed in this article. The author is also grateful to the Elementary Particle Physics Group at the Tokyo Institute of Technology for giving him the opportunity to review the developments in Liouville theory.

Funding

Open Access funding: SCOAP^3.

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